Propagation of Sound Wave in Rarefied Polyatomic Gases Based on Extended Thermodynamics

II. Transverse Wave

The basic equations are the linearized equations in the neighborhood of a reference equilibrium state:

\[ \dot{\rho} + \rho \frac{\partial v_i}{\partial x_i} = 0, \]

\[ \rho_0 \frac{\partial v_i}{\partial t} + aT_0 \frac{\partial p}{\partial x_i} + \rho_0 \frac{\partial T}{\partial x_i} - \frac{\partial S_{(\phi)}}{\partial x_i} + \frac{\partial \Pi}{\partial x_i} = 0, \]

\[ a \rho \frac{\partial \epsilon_I}{\partial x_i} + \frac{\partial \pi}{\partial x_i} \frac{\partial \pi}{\partial x_i} = 0, \]

\[ \dot{S}_{(\phi)} - 2a \rho \frac{\partial T_0}{\partial x_i} - \frac{2}{1 + c_v} \frac{\partial q}{\partial x_i} = -\frac{1}{\tau_q} S_{(\phi)}, \]

\[ \Pi + \left( \frac{2}{3} - 1 \right) \rho_0 \frac{\partial v_i}{\partial x_i} + c_v \frac{\partial q}{\partial x_i} = -\frac{1}{\tau_q}, \]

where a quantity with the subscript 0 represents the quantity in the reference equilibrium state and the summation convention is adopted. Here \( c_v \) and \( x_i \) are, respectively, the dimensionless heat capacity of state vector \( (dx/dT)_{\phi/a} \) and the position. A dot on a quantity represents material time derivative, and \( \tau_q, \tau_\Pi \) are the relaxation times which are related to the shear viscosity \( \mu \), the bulk viscosity \( \nu \) and the heat conductivity \( \kappa \), respectively.

3. Dispersion relation for transverse waves

We consider a plane harmonic wave propagating in \( x_1 \)-direction. Without loss of generality, we can assume the following type of the solution of a transverse wave:

\[ v_i = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad S_{(\phi)} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q_i = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}. \]

Note that we can easily prove that \( \rho = \rho_0, \quad T = T_0 \) and \( \Pi, \Pi \) must be zero. The solution is expressed, with frequency \( \omega \) and complex wave number \( k \), by

\[ u = we^{i(\omega t - kx_1)}, \]

where \( u, w \) and \( t \) are, respectively, the state vector \( (v_i, S_{(\phi)}, q_i) \), a constant amplitude vector and time.
The dispersion relation is obtained by
\[ -\alpha^2 \left( \frac{2 + c^2_0}{1 + c^2_0} \right) \frac{i}{\Omega \tau_\phi} \left( z^2 + 1 - \frac{1}{\Omega^2 \tau_\phi} \right) - i \frac{1 + \tau_\phi}{\Omega \tau_\phi} = 0, \]
where \( \Omega = \tau_\omega, \tau_\phi = \tau_\phi / \tau_\omega, z = k / \omega \). The phase velocity \( v_{ph} \) and the attenuation factor \( \alpha \) are expressed in terms of the solution \( z \) of the dispersion relation:
\[
\begin{align*}
    v_{ph}(\omega) &= \frac{\omega}{\text{Re}(k)} = \frac{1}{\text{Re}(z)}, \\
    \alpha(\omega) &= -\omega \text{Im}(k) = -\omega \text{Im}(z).
\end{align*}
\]

4. Typical results: the case of n-H\(_2\)

We study the dispersion relation numerically. For example, the results in the case of normal hydrogen are shown. The results for other kinds of gases will be reported at the conference. We can estimate the temperature dependence of \( \nu \) by using the experimental data on \( \mu \) and \( \kappa \) [3]. The conditions of the numerical analysis are as follows: \( T_0 = 293[K], c_0 = 2.45 \) and \( \tau_\phi = 1.46 \).

Figure 1 shows the dependence of the dimensionless phase velocity \( v_{ph}/c_0 \) and the attenuation factor \( c_0 \tau_\phi \alpha \) on the dimensionless frequency \( \Omega \), where \( c_0 \) is the (longitudinal) sound velocity in the low-frequency limit in the reference equilibrium. It can be seen that both \( v_{ph}/c_0 \) and \( c_0 \tau_\phi \alpha \) approach zero as \( \Omega \) tends to zero. When \( \Omega \) becomes large, these have nonzero values and approach some finite values as \( \Omega \) tends to infinity.

Figure 2 shows the ratio between the characteristic length \( \delta = 1/\alpha \), which is the propagation length that the amplitude becomes \( 1/e \) times, and the wavelength \( \lambda \). This ratio represents how quickly the amplitude decreases compared with the wavelength. In order to observe a vibrating character of a wave, the value of this ratio must be larger than \( O(10^6) \). It is seen from Figure 2 that the ratio has very small value in the low-frequency region and increases monotonically with the increase of \( \Omega \). The above necessary condition becomes to be satisfied when \( \Omega \) is higher than \( O(10^6) \). We conclude that \( \Omega \) should be higher than \( O(10^6) \) for experimental observations of the transverse waves.

5. Summary and concluding remarks

In this paper, the experimental condition to measure transverse waves in rarefied polyatomic gases quantitatively is obtained based on the recently-developed ET theory.

The conclusion is as follows: It is difficult to detect the transverse waves in the low-frequency region in which the NSF theory is valid. However, the transverse waves are measurable in the high-frequency region with \( \Omega \) higher than \( O(10^6) \). We expect the experiments for high-frequency transverse waves, for example, transverse waves of \( \Omega = 5.0 \) which corresponds to \( \omega \approx 0.5[\text{GHz}] \) at the pressure \( p_0 = 1000[\text{Pa}] \).

Because the wavelength of a transverse wave is very small and because the wave decays quickly, the results may play important roles in the phenomena in small scale, for example, those in nano-technology.

Acknowledgment

The authors thank Prof. Tommaso Ruggeri for valuable discussions. This work was partially supported by Japan Society of Promotion of Science (JSPS) No. 24760055 (S.T.), No. 20560054 (M.S.).

References