Analysis of guided wave which propagates pipe or pipe with fluid with attenuation
パイプを伝搬するガイド波の減衰を考慮した解析

Harumichi Sato†, and Hisato Ogiso (AIST)
佐藤治道†，小木曾久人（産総研）

1. Introduction
Cylindrical pipes are widely used in industries such as nuclear power plants and micro total analysis systems (µTAS). Nondestructive evaluation (NDE) of such pipes is therefore crucial. NDE as well as ultrasonic flowmeters can be used to characterize pipes filled with fluid. Guide wave of a hollow pipe was investigated theoretically by Gazis1, and we previously expanded on the theory proposed by Gazis for a fluid-filled pipe 2,3. Those studies were for the condition that attenuations of the pipe and the fluid are negligibly small. However, the attenuations can not be neglected in some condition such as a inspection of a spallation neutron source mercury target, and a nondestructive inspection of an erosion of the mercury container walls4 is required. Therefore, we analyzed the guided wave which propagates pipe with attenuation.

2. Theoretical analysis

Fig. 1 shows the theoretical model of a cylindrical pipe and its coordinate system (cylindrical coordinates). The author's theoretical basis is an expansion of that of hollow pipe by Gazis1. The displacement \( \mathbf{u}^{\text{solid}} \) of the pipe \( (a \leq r \leq b) \) and the displacement \( \mathbf{u}^{\text{fluid}} \) of a fluid \( (0 \leq r \leq a) \) are represented by a vector \( \mathbf{H} \) and scalar potential \( \phi_s \), \( \phi_f \) as follows.

\[
\mathbf{u}^{\text{solid}} = \nabla \phi_s + \nabla \times \mathbf{H} \quad \mathbf{u}^{\text{fluid}} = \nabla \phi_f
\] (1)

Wave equations of the potentials are as follows.

\[
v_i^2 \nabla^2 \phi_s = \frac{\partial^2 \phi_s}{\partial t^2} \quad \text{(2)}
\]

\[
v_i^2 \nabla^2 \phi_f = \frac{\partial^2 \phi_f}{\partial t^2} \quad \text{(3)}
\]

Here, \( t \) indicates time, \( v_i \), \( \nu_i \) and \( \nu_f \) represent sound velocities of longitudinal wave of pipe, transversal wave of pipe and longitudinal wave of fluid, respectively. The potentials are as follows.

\[
\phi_s = f_s(r) \cos n \theta \exp(i(k^* z - \omega t))
\]

\[
H_r = g_r(r) \sin n \theta \exp(i(k^* z - \omega t + \pi / 2))
\]

\[
H_\theta = g_\theta(r) \sin n \theta \exp(i(k^* z - \omega t + \pi / 2))
\]

\[
H_z = g_z(r) \sin n \theta \exp(i(k^* z - \omega t)) \quad \text{(3)}
\]

\[
k^* = k(1+i\eta), \quad k_f^* = k_f(1+i\eta_f)
\]

\[k_f = \frac{\omega}{V - \nu} \quad \frac{\omega}{k}
\]

\[k, \omega, n, i, \eta, \eta_f\] and \( V \) represents the wave number of the guided wave propagating in a pipe, the angular frequency, the circumferential mode parameter, the imaginary unit and an attenuation constant of the pipe, an attenuation constant of the fluid and a flow velocity of the fluid, respectively. By eq.(2) and (3), below equations are obtained.

\[
f_s = A_s Z_n(\alpha r) + B_s W_n(\alpha r)
\]

\[
g_s = A_s Z_n(\beta r) + B_s W_n(\beta r)
\]

\[
2g_1 = (g_s - g_0) = 2A_s Z_{n+1}(\beta r) + 2B_s W_{n+1}(\beta r) \quad \text{(4)}
\]

\[
2g_2 = (g_s + g_0) = 2A_s Z_{n-1}(\beta r) + 2B_s W_{n-1}(\beta r)
\]

\[
f_f = A_f Z_n(\zeta r)
\]

\[
\alpha^2 = \omega^2 / \nu_i^2 - k^2
\]

\[
\beta^2 = \omega^2 / \nu_f^2 - k_f^2 \quad \text{(5)}
\]

\[J_n, Y_n, I_n, K_n\] are the Bessel function of the
first kind, the Bessel function of the second kind, the modified Bessel function of the first kind and the modified Bessel function of the second kind, respectively. \( Z_n, W_n, \alpha_n, \beta_n \) and \( \chi_n \) are show in Table 1-3. Each argument of the Bessel functions become a complex number when \( \eta \) or \( \eta_f \) is not zero. By the property of the gauge invariance, any one of the three potentials \( g_i \) \((i = 1, 2, \text{or } 3)\) can be set to zero. Setting \( g_2 = 0 \) yields

\[
g_r = -g_0 = g_1
\]

By eq.(1), (3) and (6), the displacements are as follows.

\[
\begin{align*}
\mathbf{u}_{r, \text{solid}} &= \mathbf{u}_{r, \text{fluid}}, \\
\sigma_{r, \text{solid}} &= \sigma_{r, \text{fluid}} = 0 \quad \text{at } r = a, \\
\sigma_{r, \text{solid}} &= \sigma_{r, \text{fluid}} = 0 \quad \text{at } r = b
\end{align*}
\]

The boundary conditions are as follows.

\[
\begin{align*}
\mathbf{u}_{r, \text{solid}} &= \mathbf{u}_{r, \text{fluid}}, \\
\sigma_{r, \text{solid}} &= \sigma_{r, \text{fluid}} = 0 \quad \text{at } r = a, \\
\sigma_{r, \text{solid}} &= \sigma_{r, \text{fluid}} = 0 \quad \text{at } r = b
\end{align*}
\]

\( \sigma_{r, \text{solid}} \) and \( \sigma_{r, \text{fluid}} \) are the stress tensors of the pipe and fluid, respectively. They are obtained by displacements and solid’s and fluid’s densities \((\rho_s \text{ and } \rho_f)\). By eq. (7) and (8), a homogeneous systems of linear equations is obtained.

\[
[c_{ij}] \mathbf{x} = 0
\]

\[
\mathbf{x} = (A_1, A_2, A_3, B_1, B_2, B_3, A_4)
\]

\([c_{ij}]\) is a 7×7 matrix, and \(c_{ij}\) are similar to our previous result apart from \( k \) and \( k_f \). All \( k \) and \( k_f \) of \( c_{ij} \) in refs. 2 and 3 are replaced by \( k^* \) and \( k_f^* \), respectively. For example, some \( c_{ij} \) s are shown below.

\[
\begin{align*}
c_{12} &= k^* aZ_{n+1}(\beta_1a), \\
c_{17} &= [\eta n Z_n(\chi_n(\alpha) + \chi_n \lambda_2 a Z_{n+1}(\chi_n(\alpha)))] \\
&\times \exp(ik_f^* z - \omega t) / \exp(ik^* z - \omega t), \\
c_{21} &= [2(na - 1)(\beta^2 - k^2)(\alpha^2)] Z_n(\alpha^2a) \\
&+ 2 \lambda_2 \alpha a Z_{n+1}(\alpha^2a), \\
c_{27} &= [\rho_f \omega^2 a Z_n(\chi_n(\alpha)/(|\rho_f|^2))] \\
&\times \exp(ik_f^* z - \omega t) / \exp(ik^* z - \omega t)
\end{align*}
\]

A nontrivial solution is obtained when the determinant of \([c_{ij}]\) is zero.

\[
\det[c_{ij}] = 0
\]

Because eq (11) contains the frequency \((f = \omega/2\pi)\) and the phase velocity \((V)\), the dispersion curves are obtained.

### Table 1 Parameters for \( \alpha \)

<table>
<thead>
<tr>
<th>( \alpha_i )</th>
<th>( \lambda_n )</th>
<th>( Z_n(\alpha_i^r) )</th>
<th>( W_n(\alpha_i^r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re(( \alpha_i^2 )) &gt; 0</td>
<td>( \alpha_i )</td>
<td>1</td>
<td>( J_n(\alpha_i^r) )</td>
</tr>
<tr>
<td>Re(( \alpha_i^2 )) &lt; 0</td>
<td>( \alpha_i/i )</td>
<td>-1</td>
<td>( I_n(\alpha_i^r) )</td>
</tr>
</tbody>
</table>

### Table 2 Parameters for \( \beta \)

<table>
<thead>
<tr>
<th>( \beta_i )</th>
<th>( \lambda_\beta )</th>
<th>( Z_n(\beta_i^r) )</th>
<th>( W_n(\beta_i^r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re(( \beta_i^2 )) &gt; 0</td>
<td>( \beta_i )</td>
<td>1</td>
<td>( J_n(\beta_i^r) )</td>
</tr>
<tr>
<td>Re(( \beta_i^2 )) &lt; 0</td>
<td>( \beta_i/i )</td>
<td>-1</td>
<td>( I_n(\beta_i^r) )</td>
</tr>
</tbody>
</table>

### Table 3 Parameters for \( \chi \)

<table>
<thead>
<tr>
<th>( \chi_i )</th>
<th>( \lambda_\chi )</th>
<th>( Z_n(\chi_i^r) )</th>
<th>( W_n(\chi_i^r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re(( \chi_i^2 )) &gt; 0</td>
<td>( \chi_i )</td>
<td>1</td>
<td>( J_n(\chi_i^r) )</td>
</tr>
<tr>
<td>Re(( \chi_i^2 )) &lt; 0</td>
<td>( \chi_i/i )</td>
<td>-1</td>
<td>( I_n(\chi_i^r) )</td>
</tr>
</tbody>
</table>

### 3. Discussions and Conclusions

We obtained anyitational result of the guided wave which propagates pipe or pipe with fluid with attenuation. As a sample, the determinant of \([c_{ij}]\) is plotted in Fig. 2. We can see two \( V \) s in Fig. 2.

### Acknowledgment

This work was supported by MEXT/JSPS KAKENHI Grant Number 23561021.

### References