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Acoustic Resonance of One-dimensional Nonlinear Elastic String
一次元非線形弾性体の共鳴振動解析

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1. Introduction

In 1971, H.H. Demarest demonstrated that the second order elastic constants C_{ijkl} of a rectangular parallelepiped shape sample can be obtained from inverse analysis to free vibration resonance frequencies.[1] Following to Demarest, several researchers contributed further development of the method.[2] Nowadays it is called resonant ultrasound spectroscopy (RUS).[3] The RUS technique has been applied succesfully to several kinds of materials. However, to the authors' understanding, the theory of RUS is still in its progress and further developments are required. The purpose of this work is to extending the theory into a general framework of nonlinear elasticity. For the sake of simplicity, we employ an one-dimensional elastic string model. While the model is rather simple, yet it includes theoretical developments that should be required on a formulation in higher dimensions.

2. Variational Formulation

Let us define the one-dimensional nonlinear elastic string by $\Omega = \{x \in \mathbb{R} \mid x \in [-1, 1]\}$. The Ω undergoes a pure longitudinal deformation $u = u(x,t)$. Then, the kinetic energy density T due to the deformation u is written as

$$T = \frac{1}{2}\rho u_t^2.$$

Where the ρ is mass-density of the Ω and assumed to be uniform when the deformation is zero; $u = 0$ for all x in Ω. The $u_t = \partial u/\partial t$ represents the deformation velocity. On the other hand, the strain energy density due to the deformation u can be expressed by a Taylor series expansion such that

$$E = \frac{1}{2}k_1 u_x^2 - \frac{1}{6}k_2 u_x^3 + O(u_x^4).$$

Where $k_1 = C_{1111}$ and $k2 = -(3C_{1111}+C_{111111}) > 0$. The $u_x = \partial u/\partial x$ denotes deformation gradient. **Fig. 1** plots the strain energy density E with repect to the deformation gradient u_x. The domain of the function E is set to be $u_x < 2k_1/k_2$.

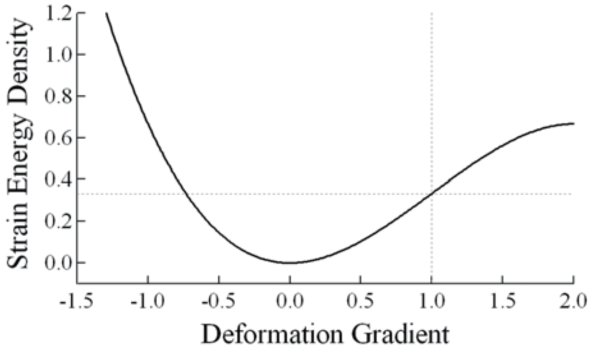

Figure 1. Strain energy density E plotted as a function of the deformation gradient u_x. The horizontal and vertical axes are normalized by the k_1/k_2 and k_1^3/k_2^2.

From the kinetic and strain energy densities, we can express the lagrangian density of the elastic string as $L = T - E$. Integration of the L on the fixed domain $x \in [-1, 1]$ and on a certain time interval $t \in [t_0, t_1]$ yields a functional of the action integral

$$I[u] = \int_{t_0}^{t_1}\int_{-1}^{1} L(u_t, u_x)\ dxdt.$$

According to the principle of least action, an actual deformation u must satisfy the stationary condition; $\delta I = 0$. This is the variational formulation for the acoustic resonance of Ω. Note that the condition $\delta I = 0$ poses a *variable domain problem* since the domain on t is not fixed in general.

3. Direct Analysis by the Ritz Method

In order to solve the variational problem we employ a direct analysis which is based on the Ritz method. More precisely, we expand the deformation function u by complex Fourier series such that,

$$u = \sum_{n=-N}^{N}\left[\frac{a_{n,0}}{2} + \sum_{m=1}^{M}\{a_{n,m}\cos m\pi x + b_{n,m}\sin(2m-1)\pi x\}\right] \times e^{in\omega t}$$

The ω represents the resonance frequency. The

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function u satisfies the *natural boundary conditions* which are required at end points of the domains. In order to prevent a trivial solution, we impose a subsidiary condition; $\|u\|_{L^2} = \text{const}$. Inserting the u into the I and taking stationary conditions given by Ritz, it end up with a set of nonlinear simultaneous equations. This problem has been solved numerially by using the Newton method.

4. Results and Discussion

Fig. 2 show the amplitude (or L^2 norm) dependence of resonance frequencies obtained from the first three vibration modes. At the lowest amplitude, ω are close to the linear case solutions; $\omega^{1st.} = \pi/2$, $\omega^{2nd.} = \pi$ and $\omega^{3rd..} = 3\pi/2$. However, due to the nonlinearities of the system, they show monotonic decreasing with increasing in amplitude. To our surprise, these behaviors are similar to the temperature dependence of resonance frequency observed in low temperature RUS experiments.

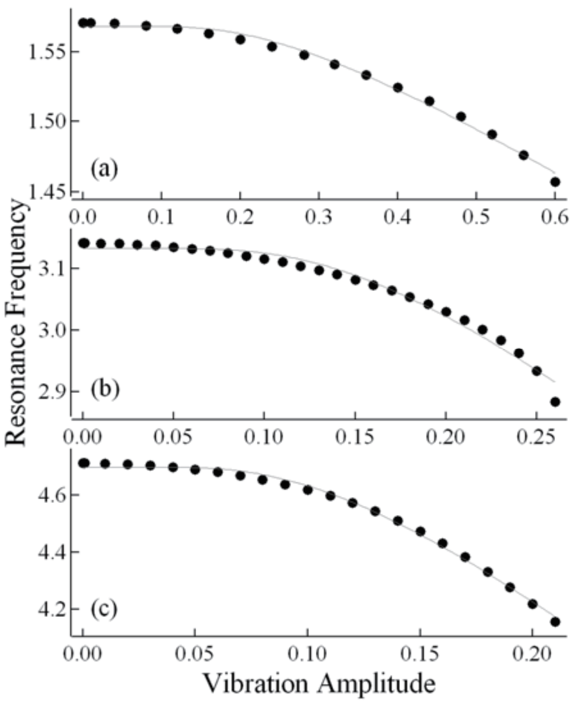

Figure 2. Amplitude dependence of resonance frequency for the first three resonant modes.

Fig. 3 show time dependence of the longitudinal deformation u obtained from the first three vibration modes. While overall vibration patterns are close to those of the linear ones, we could see some prominent nonlinear features. First of all, the vibration patterns are no more symmetric with respect to time. For instance, deformation amplitude of the first mode at x = 1 is 1.01 while those after the half period is only -0.88. The same apply to the second and third modes. The second mode lost the symmetry in space either; there is no mirror symmetry on the x axis in any t. Secondly, there is no specific time t^* at which all deformation in Ω vanishes. This is also a crucial difference between the previous linear and present nonlinear system. The last feature we would like to mention is seen at x = ±1/2 in the second mode and x = ±2/3 in the third. In the nonlinear system, these points are no longer the nodal point and slightly oscillating with the frequencies of 2ω.

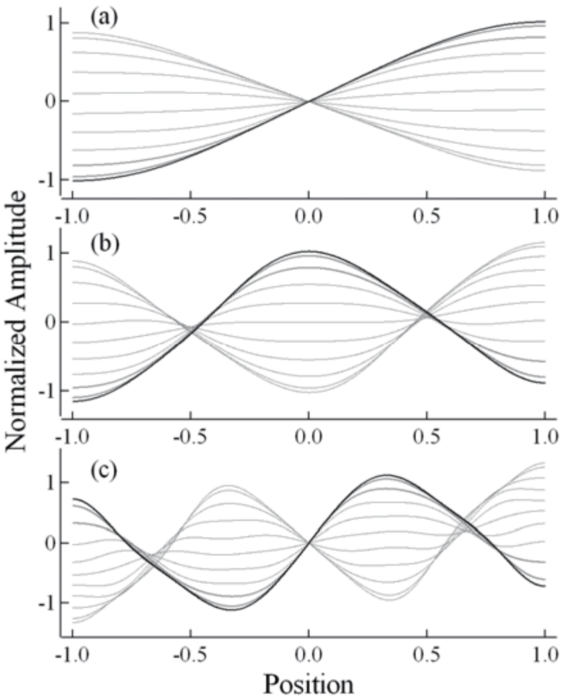

Figure 3. Resonance vibrations for the first three resonant modes.

5. Conclusions

In this study, we have investigated free vibration acoustic resonance of one-dimensional nonlinear elastic string Ω. Resonant states of the string have been formulated within a framework of the calculus of variation and its stationary conditions have been solved numerically by using a direct analysis which is based on the Ritz method. The Ω showed prominent nonlinear features such as amplitude dependence of resonance frequencies, symmetry breaking in the vibration modes, and excitation of high frequency modes.

References

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